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# A Generalized Field Theory. II. Linearized Field Equations

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A thorough examination of the generalized field theory, formulated by the same authors in a previous paper, is being carried out in the absence of feedback effects. The results obtained are found to be in complete agreement with those of linear field theories of gravity and electromagnetism. Strict functions, which serve as indicators of the strength of the two fields, are being identified. This study reveals also two interesting results: the first is the classification of tetrad vector fields used, the second is the definite appearance of a mutual interaction between gravitational and electromagnetic fields.

## 1. INTRODUCTION

In a previous paper (Mikhail and Wanas (1977) referred to hereafter as I), the authors were able to formulate a generalized field theory using a tetrad space. The field equations of this theory were found to have the same strength as those of general relativity. In that paper the gravitational and electromagnetic features of the new theory are explored. Various quantities of physical interest are identified with the elements of the geometrical structure used in the formalism.

We think that it is of great interest to reveal all possible links between the new theory and the classical field theories of gravity and electromagnetism. This can be done by applying the new theory to weak fields. In this case it is necessary to compare the results of the new theory with those of general relativity and Maxwell's theory under similar conditions. It is also of



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interest to point out that this study has helped in revealing the mutual interaction between electromagnetism and gravitational fields.

In Section 2 we discuss a method for expanding the derived elements of the field, in ascending orders of magnitude. Then, expressing the field equations in the same manner, we are able to get a linearized set of field equations. In Section 3 we study the solution of these linearized field equations. Detailed discussion of the results obtained is presented in Section 4, and a final conclusion is given in Section 5.

## 2. THE METHOD OF LINEARIZATION

The field equations obtained in (I) are nonlinear in the field variables. This nonlinearity is thought to give rise to a feedback effect, in addition to the original source of the field. Thus in order to compare the new field equations with the classical equations of gravity and electromagnetism, it is customary to remove this factor at first. That is to put these equations into a linearized form.

In the new theory, the field variables are the tetrad vectors $\underset{i}{\lambda}\mu (i, \mu = 0, 1, 2, 3)$. The values of these vectors in the Galilean inertial frame of special relativity are given by

$$\underset{i}{\lambda}\mu = \delta_{i\mu} \tag{1}$$

where $\delta_{i\mu}$ are the Kronecker deltas. Hence, to get a space which differs slightly from the flat space based on (1), we may take the tetrad vectors in the form

$$\underset{i}{\lambda}\mu = \delta_{i\mu} + \epsilon \underset{i}{h}\mu \tag{2}$$

where $\epsilon \underset{i}{h}\mu$ represents a perturbation term, $\epsilon$ is a parameter, and $\underset{i}{h}\mu$ are functions of the coordinates. The parameter $\epsilon$ is assumed to be of small magnitude compared with unity.

In deriving the new field equations, we have assumed that all physical fields are generated by the tetrad vectors $\underset{i}{\lambda}\mu$ and their derivatives. Hence, we can express each derived element of the field in terms of different orders of the parameter $\epsilon$ in the following manner. If $M$ is any derived member of the tetrad, we can express it in the form

$$M = \sum_r \epsilon^r \overset{(r)}{M} \tag{3}$$

where $r$ $(=0,1,2,3,\ldots)$ is the power of $\epsilon$ and $\overset{(r)}{M}$ is the coefficient of $\epsilon^r$. For example, the symmetric tensor $g_{\mu\nu}(=\underset{i}{\lambda}\mu\underset{i}{\lambda}\nu)$ using (2), is found to be

$$g_{\mu\nu}=\delta_{\mu\nu}+\epsilon y_{\mu\nu}+\epsilon^2 \underset{i}{h}_{\mu}\underset{i}{h}_{\nu} \tag{4}$$

where

$$y_{\mu\nu}=\underset{\mu}{h}_{\nu}+\underset{\nu}{h}_{\mu} \tag{5}$$

Thus comparing (4) with (3) we get $\overset{(1)}{g}_{\mu\nu}=y_{\mu\nu}$, $\overset{(2)}{g}_{\mu\nu}=\underset{i}{h}_{\mu}\underset{i}{h}_{\nu}$.

The field equations derived in (I) have the form

$$E_{\mu\nu}=0 \tag{6}$$

where $E_{\mu\nu}$ is a second-order nonsymmetric tensor. This tensor can be written explicitly, in terms of the tetrad tensors, as

$$\begin{aligned} E_{\mu\nu} \overset{\text{def}}{=} g_{\mu\nu}L-2L_{\mu\nu}-2g_{\mu\nu}C^{\alpha}_{\underset{-}{}|\alpha}-2C_{\mu}C_{\nu}-2g_{\mu\alpha}C^{\epsilon}\Lambda^{\alpha}_{\epsilon\nu} \\ +2C_{\underset{+}{\nu}|\mu}-2g^{\alpha\beta}\Lambda_{\underset{+++}{\mu\nu\beta}|\alpha} \end{aligned} \tag{7}$$

where the $(+)$ and the $(-)$ derivatives appearing in this expression are defined in the usual manner with respect to the nonsymmetric connection $\Gamma^{\alpha}_{\mu\nu}(=\underset{i}{\lambda}^{\alpha}\underset{i}{\lambda}_{\mu,\nu})$ [the same connection used by Einstein (1929) and Robertson (1932)]; and

$$\begin{aligned} L &\overset{\text{def}}{=} g^{\mu\nu}L_{\mu\nu} \\ L_{\mu\nu} &\overset{\text{def}}{=} \Lambda^{\alpha}_{\beta\mu}\Lambda^{\beta}_{\alpha\nu}-C_{\mu}C_{\nu} \\ \Lambda^{\alpha}_{\mu\nu} &\overset{\text{def}}{=} \Gamma^{\alpha}_{\mu\nu}-\Gamma^{\alpha}_{\nu\mu} \\ C_{\mu} &\overset{\text{def}}{=} \Lambda^{\alpha}_{\mu\alpha} \\ \Lambda_{\alpha\mu\nu} &\overset{\text{def}}{=} g_{\alpha\beta}\Lambda^{\beta}_{\mu\nu} \end{aligned} \tag{8}$$

So using (2), we can express each of the tensors on the right-hand side of (8), and the field equations (6), in ascending powers of the parameter $\epsilon$ in the manner indicated by (3).

## 3. LINEARIZED FIELD EQUATIONS

Using the field equations (6), and confining ourselves to linear terms only ($r=1$), we get the following results:

**The Symmetric Part.** The linearized equations arising from the symmetric part of the field equations (6) can be written, using (5), in the following form:

$$\Box^2 y_{\mu\nu} = \left(y_{\mu\alpha,\alpha} - \tfrac{1}{2} y_{\alpha\alpha,\mu}\right)_{,\nu} + \left(y_{\nu\alpha,\alpha} - \tfrac{1}{2} y_{\alpha\alpha,\nu}\right)_{,\mu} \tag{9}$$

where $\Box^2$ is the four-dimensional D'Alembertian operator. Equation (9) is identical with that obtained from general relativity in the case of weak field. So following Weyl's method (cf. Adler et al., 1975), we get the following solution for (9):

$$y_{\mu\nu} = \phi_{\mu,\nu} + \phi_{\nu,\mu} \tag{10}$$

where $\phi_\mu$ is an arbitrary four-component function of the coordinates satisfying the relation

$$\Box^2 \phi_\mu = \tau_\mu = \left(y_{\mu\alpha,\alpha} - \tfrac{1}{2} y_{\alpha\alpha,\mu}\right) \tag{11}$$

For any arbitrary solution $X_{\mu\nu}$, of the field equations (9), we can obtain the associated Weyl solution $Y_{\mu\nu}$. Besides, since equations (9) are linear, the difference between these two solutions will give a third one, namely,

$$\overset{*}{Y}_{\mu\nu} = X_{\mu\nu} - Y_{\mu\nu} \tag{12}$$

which satisfies the relation

$$\Box^2 \overset{*}{Y}_{\mu\nu} = 0 \tag{13}$$

This last equation represents a wave equation, which may be of some interest.

**The Skew Part.** Treating the skew part of (6) in a similar manner, we get the equation

$$\overset{(1)}{\xi}_{\mu\nu} = \overset{(1)}{C}_{\nu,\mu} - \overset{(1)}{C}_{\mu,\nu} \tag{14}$$

where $\xi_{\mu\nu}$ is a skew tensor defined by

$$\xi_{\mu\nu} = \gamma_{\underset{++}{\mu\nu}}{}^{\epsilon}{}_{+|\epsilon}$$

and $\gamma_{\mu\nu\alpha}$ is the world tensor corresponding to the Ricci coefficients of rotation (Cf. Eisenhart 1927). But the skew part of (6) can be written explicitly, according to (3), in the form

$$\epsilon^r \overset{(r)}{F}_{\mu\nu} = \epsilon^r \overset{(r)}{C}_{\mu,\nu} - \epsilon^r \overset{(r)}{C}_{\nu,\mu} \tag{15}$$

where

$$F_{\mu\nu} \overset{\text{def}}{=} Z_{\mu\nu} - \xi_{\mu\nu}$$

and

$$Z_{\mu\nu} \overset{\text{def}}{=} \eta_{\mu\nu} + \zeta_{\mu\nu} \tag{16}$$

The right-hand side of (16) are skew tensors defined by

$$\zeta_{\mu\nu} \overset{\text{def}}{=} C_\alpha \gamma_{\mu\nu}{}^{\alpha}$$

$$\eta_{\mu\nu} \overset{\text{def}}{=} C_\alpha \Lambda^{\alpha}{}_{\mu\nu}$$

Now from (14) and (15) we can see that $\eta_{\mu\nu}, \zeta_{\mu\nu}$ have no linear terms. This result will be discussed later.

The current vector density was defined in (I) as

$$\mathfrak{J}^{\mu} \overset{\text{def}}{=} \mathfrak{F}^{\mu\nu}{}_{,\nu} \tag{17}$$

where

$$\mathfrak{F}^{\mu\nu} \overset{\text{def}}{=} \overset{*}{\lambda} F^{\mu\nu} \qquad \left( \overset{*}{\lambda} \overset{\text{def}}{=} \| \underset{i}{\lambda}_{\mu} \| \right) \tag{18}$$

It is to be noted that, in the linearized theory, raising and lowering of indices are carried out using Kronecker deltas; and the determinant $\overset{*}{\lambda}$ will have the value

$$\overset{*}{\lambda}=1+\epsilon\left(\underset{0}{h}_{0}+\underset{1}{h}_{1}+\underset{2}{h}_{2}+\underset{3}{h}_{3}\right) \tag{19}$$

So the vector density (17) will be reduced to the vector $J^{\mu}$ itself, and using (14), we can write (17) in the following linearized form:

$$\overset{(1)}{J}_{\mu}=\overset{(1)}{C}_{\mu,\nu\nu}-\overset{(1)}{C}_{\nu,\nu\mu} \tag{20}$$

Contracting (9) and using (5), we get

$$\underset{\alpha}{h}_{\nu,\alpha\nu}=\underset{\alpha}{h}_{\alpha,\nu\nu} \tag{21}$$

But expanding $c_{\mu}$ using (2), we get

$$\overset{(1)}{C}_{\mu,\mu}=\left(\underset{\alpha}{h}_{\mu,\alpha\mu}-\underset{\alpha}{h}_{\alpha,\mu\mu}\right)$$

So using (21), we get

$$\overset{(1)}{C}_{\mu,\mu}=0 \tag{22}$$

Substituting from (22) into (20), we get

$$\Box^{2}\overset{(1)}{C}_{\mu}=\overset{(1)}{J}_{\mu} \tag{23}$$

## 4. DISCUSSION

(a) The previous treatment shows that some results of the new theory, in its linearized form, are classically known.

The so-called material-energy tensor $T_{\mu\nu}$ of the new theory has been found to satisfy the relation

$$T^{\mu\nu}{}_{;\nu}=0$$

where

$$T_{\mu\nu} \overset{\text{def}}{=} g_{\mu\nu} \wedge + \tilde{\omega}_{\mu\nu} - \sigma_{\mu\nu}, \qquad \wedge \overset{\text{def}}{=} \tfrac{1}{2}(\sigma - \tilde{\omega}) \tag{24}$$

The tensors

$$\tilde{\omega}_{\mu\nu}\left(=\gamma^{\epsilon}_{\mu\alpha}\gamma^{\alpha}_{\epsilon\nu} + \gamma^{\epsilon}_{\nu\alpha}\gamma^{\alpha}_{\epsilon\mu}\right), \qquad \sigma_{\mu\nu}\left(=\gamma^{\epsilon}_{\alpha\mu}\gamma^{\alpha}_{\epsilon\nu}\right)$$

are symmetric. In the first approximation, covariant differentiation reduces to ordinary partial differentiation, and thus (24) can be reduced to the classical form of the law of conservations of matter–energy.

Also in the new theory the current vector density $\mathfrak{T}^{\mu}$ has been found to satisfy the identity

$$\mathfrak{T}^{\mu}{}_{,\mu} \equiv 0 \tag{25}$$

This identity will reduce to the same form as the classical law of conservation of charge.

In the case of static weak fields, equation (9) can be reduced to the classical Laplace's equation for gravity when $\mu = \nu = 0$; also equation (23), under the same conditions, will be reduced to Poisson's equation for a charge distribution (with a special choice of units).

Taking as usual the symmetric tensor $g_{\mu\nu}$ as representing the gravitational potential, the solution (13) shows that the first-order variations in the gravitational potential are propagated in the form of waves with the velocity of light. Also equation (23) will reduce in empty space to a wave equation showing that the vector $c_{\mu}$ is also propagated with the fundamental velocity. The vector $c_{\mu}$ has been identified (I) with the electromagnetic potential. The results discussed above are exactly similar to those obtained before using general relativity and Maxwell's theory.

(b) We think that there are strong reasons to consider the scalar function $\wedge$, given by (24), as an indicator showing the strength of the gravitational field.

We can easily see from (24), by contraction, that

$$\wedge = \tfrac{1}{2}T$$

Thus $T$ vanishes with $\wedge$. We also note that the two tensors $\tilde{\omega}_{\mu\nu}$ and $\sigma_{\mu\nu}$ when expressed in powers of $\epsilon$ will start with terms in $\epsilon^2$. Hence the scalar $\wedge$ will consequently involve only terms of the second and higher orders in $\epsilon$. This means that, in the new theory, if we keep only first-order terms in $\epsilon$ in the field potentials, we get $T = 0$, and so also $\wedge = 0$ as a direct consequence.

Similarly, as the two tensors $\eta_{\mu\nu}$, $\zeta_{\mu\nu}$ start with the terms in $\epsilon^2$, we can take the tensor $Z_{\mu\nu}(=\eta_{\mu\nu}+\zeta_{\mu\nu})$ as an indicator showing the strength of the electromagnetic field, such that its nonvanishing means that the electromagnetic field is strong.

(c) The above study provides a means for getting some important physical characteristics specifying the field under consideration directly from the geometrical elements of the tetrad space used. Namely, we can get, off hand, a clear idea about the strength of the gravitational field as well as the electromagnetic field represented by a certain tetrad space before the actual study of the field equations. As a result of this treatment we can specify some distinct classes of gravitational fields (denoted by the letter G) and electromagnetic fields (denoted by the letter F) according to Table I. Possible combinations of the two fields which may be of physical interest are classified into two main groups:

The first group: F0G0, F0GI, F0GII, FIGII.

The second group: F0GIII, FIGIII, FIIGII, FIIGIII.

Models belonging to the first group are expected to show full agreement with the classical field theories of gravitation and electromagnetism. Deviations from classical field theories will only appear when using models belonging to the second group.

More recently, in a trial to describe strong gravitational fields, Møller (1978) has established a generalized field theory based on a tetrad space. He

**Table I**

| Indicator | Field represented | Type |
|---|---|---|
| $F_{\mu\nu}=0$ | No electromagnetic field | F0 |
| $F_{\mu\nu}\neq 0$<br>$Z_{\mu\nu}=0$ | Electromagnetic field, not strong | FI |
| $F_{\mu\nu}\neq 0$<br>$Z_{\mu\nu}\neq 0$ | Strong electromagnetic field | FII |
| $R^{\alpha}_{\cdot\mu\nu\sigma}=0$ [a] | No gravitational field | G0 |
| $T_{\mu\nu}=0$<br>$\Lambda=0$ | Gravitational field in empty space, not strong | GI |
| $T_{\mu\nu}\neq 0$<br>$\Lambda=0$ | Gravitational field within a material distribution, not strong | GII |
| $T_{\mu\nu}\neq 0$<br>$\Lambda\neq 0$ | Strong gravitational field within a material distribution | GIII |

[a] The tensor $R^{\alpha}_{\cdot\mu\nu\sigma}$ is defined in (I), equation (4.2).

failed to obtain results different from those obtained by using orthodox general relativity. This is due to the tetrad space he used rather than to his field equations. In fact he used in his application a tetrad space of the type denoted above by F0GI. We believe that deviations from general relativity, if any, can only be discovered if tetrads of the second group are used in the testing application.

(d) A final interesting result of the above study, which may appear for the first time, is the mutual interaction between the gravitational and electromagnetic fields. This is shown in the condition (22) to be satisfied by the electromagnetic potential $c_\mu$, as a direct consequence of the vanishing of the scalar curvature $\overset{(1)}{R}=0$, in the first approximation. As the scalar curvature $R$ is usually considered to be of a gravitational nature, condition (22) expresses the effect of the gravitational field induced on the electromagnetic potential. It is of interest to note that a condition similar to (22) is usually assumed to hold in Maxwell's classical theory, in order to remove the ambiguity in the field equations, while it presents itself, here, quite naturally.

Another interesting feature of this interaction can be seen into the following. Expanding $c_\mu$ in terms of $\epsilon$, up to the first power only, we get

$$\overset{(1)}{c}_\mu = y_{\alpha\mu,\alpha} - \tfrac{1}{2} y_{\alpha\alpha,\mu} - \underset{\mu}{h}_{\alpha,\alpha} \tag{26}$$

The first two terms on the right-hand side of (26) represent the gravitational contribution to the electromagnetic potential, since $y_{\mu\nu}$ is the variation in the gravitational potential. It is clear that this contribution will be very limited in the case of gravitational fields which are not strong (GI, GII). Thus this new phenomenon can only be tested in the case of strong gravitational fields.

On the other hand, substituting from (26) into (9) we get

$$\Box^2 y_{\mu\nu} = \overset{(1)}{c}_{\mu,\nu} + \overset{(1)}{c}_{\nu,\mu} + \underset{\mu}{h}_{\alpha,\alpha\nu} + \underset{\nu}{h}_{\alpha,\alpha\mu} \tag{27}$$

The first two terms on the right-hand side of (27) represent the contribution of the electromagnetic field to the gravitational field. This type of contribution is not strictly new. A similar type of contribution appeared in studies using the Einstein–Maxwell field equations.

## 5. CONCLUSION

The above study shows that the new theory, in its linearized form, covers the domain of classical linear field theories of gravitation and electromagnetism. The new theory has given rise to almost all the classical known results. Besides, using the new theory, we get two more new results, which are of physical interest. Firstly, we can determine the strength of the field represented by a particular tetrad space. Secondly, we can describe, for the first time, in a definite way the mutual interaction between gravitational and electromagnetic fields.

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